PARAMETRISED STRICT DEFORMATION QUANTIZATION OF C^*-BUNDLES AND HILBERT C^*-MODULES

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ABSTRACT. In this paper, we review the parametrised strict deformation quantization of C^*-bundles obtained in a previous paper, and give more examples and applications of this theory. In particular, it is used here to classify H_3-twisted noncommutative torus bundles over a locally compact space. This is extended to the case of general torus bundles and their parametrised strict deformation quantization. Rieffel's basic construction of an algebra deformation can be mimicked to deform a monoidal category, which deforms not only algebras but also modules. As a special case, we consider the parametrised strict deformation quantization of Hilbert C^*-modules over C^*-bundles with fibrewise torus action.

Dedicated to Alan Carey, on the occasion of his 60^{th} birthday

INTRODUCTION

Parametrised strict deformation quantization of C^*-bundles was introduced by the authors in an earlier paper [9], as a generalization of Rieffel's strict deformation quantization of C^*-algebras [18, 19]. The particular version of Rieffel's theory that was generalized in [9] was due to Kasprzak [13] and Landstad [14, 15]. The results in [9] were used to classify noncommutative principal torus bundles as defined by Echterhoff, Nest, and Oyono-Oyono [6], as parametrised strict deformation quantizations of principal torus bundles. These arise as special cases of the continuous fields of noncommutative tori that appear as T-duals to spacetimes with background H-flux in [16, 17].

We review in §2, the construction of the parametrized deformation quantization from our earlier paper [9] and give more examples in sections 2, 5, and applications in section 3, of this construction. For example, we generalize the main application of our results in [9] (as well as those in [6]). More precisely, suppose that $A(X)$ is a C^*-bundle over a locally compact space X with a fibrewise action of a torus T, and that $A(X) \rtimes T \cong \mathrm{C}T(X, H_3)$, where $\mathrm{C}T(X, H_3)$ is a continuous trace algebra with spectrum X and Dixmier-Douady class $H_3 \in H^3(X; \mathbf{Z})$. We call such C^*-bundles, H_3-twisted NCPT bundles over X. Our first main result is that any H_3-twisted NCPT bundle $A(X)$ is equivariantly Morita equivalent to the parametrised deformation quantization of the continuous trace algebra

$$\mathrm{C}T(Y, q^*(H_3))_\sigma,$$

where $q : Y \to X$ is a principal torus bundle with Chern class $H_2 \in H^2(X; H^1(T; \mathbf{Z}))$, and $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ a defining deformation such that $[\sigma] = H_1 \in H^1(X; H^2(T; \mathbf{Z}))$. This enables us to prove in section 4 that the continuous trace algebra

$$\mathrm{C}T(X \times T, H_1 + H_2 + H_3),$$

with Dixmier-Douady class $H_1 + H_2 + H_3 \in H^3(X \times T; \mathbf{Z})$, $H_j \in H^j(X; H^{3-j}(T; \mathbf{Z}))$, has an action of the vector group V that is the universal cover of the torus T, and covering the V-action on $X \times T$. Moreover the crossed product can be identified up to T-equivariant Morita equivalence,

$$\mathrm{C}T(X \times T, H_1 + H_2 + H_3) \rtimes V \cong \mathrm{C}T(Y, q^*(H_3))_\sigma.$$

That is, the T-dual of $(X \times T, H_1 + H_2 + H_3)$ is the parametrised strict deformation quantization of $(Y, q^*(H_3))$ with deformation parameter σ, $[\sigma] = H_1$. From this we obtain the explicit dependence of the K-theory of $\mathrm{C}T(Y, q^*(H_3))_\sigma$ in terms of the deformation parameter.

In section 6, we extend this to the case of general torus bundles and their noncommutative parametrised strict deformation quantizations. It has proved a useful principle that deformation of an algebraic structure should be viewed within the context of a deformation of an appropriate category, see, for example, [2, Introduction]. Following this philosophy, we show in Section 7 how Rieffel deformations can be regarded as monoidal functors which allow us to deform modules as well as algebras. In the last section, we construct the parametrised strict deformation quantization of Hilbert C^*-modules over C^*-bundles with fibrewise torus action directly.

1. C^*-bundles and fibrewise smooth $*$-bundles

We begin by recalling the notion of C^*-bundles over X and then introduce the special case of H_3-twisted noncommutative principal bundles. Then we discuss the fibrewise smoothing of these, which is used in parametrised Rieffel deformation later on.

Let X be a locally compact Hausdorff space and let $C_0(X)$ denote the C^*-algebra of continuous functions on X that vanish at infinity. A C^*-*bundle* $A(X)$ over X in the sense of [6] is exactly a $C_0(X)$-algebra in the sense of Kasparov [12]. That is, $A(X)$ is a C^*-algebra together with a non-degenerate $*$-homomorphism

$$\Phi : C_0(X) \to ZM(A(X)),$$

called the *structure map*, where $ZM(A)$ denotes the center of the multiplier algebra $M(A)$ of A. The *fibre* over $x \in X$ is then $A(X)_x = A(X)/I_x$, where

$$I_x = \{\Phi(f) \cdot a;\, a \in A(X) \,\text{and}\, f \in C_0(X) \,\text{such that}\, f(x) = 0\},$$

and the canonical quotient map $q_x : A(X) \to A(X)_x$ is called the *evaluation map* at x.

Note that this definition does not require local triviality of the bundle, or even for the fibres of the bundle to be isomorphic to one another.

Let G be a locally compact group. One says that there is a *fibrewise action* of G on a C^*-bundle $A(X)$ if there is a homomorphism $\alpha : G \longrightarrow \mathrm{Aut}(A(X))$ which is $C_0(X)$-linear in the sense that

$$\alpha_g(\Phi(f)a) = \Phi(f)(\alpha_g(a)), \qquad \forall g \in G,\, a \in A(X),\, f \in C_0(X).$$

This means that α induces an action α^x on the fibre $A(X)_x$ for all $x \in X$.

The first observation is that if $A(X)$ is a C^*-algebra bundle over X with a fibrewise action α of a *Lie group* G, then there is a canonical *smooth $*$-algebra bundle* over X. We recall its definition from [3]. A vector $y \in A(X)$ is said to be a *smooth vector* if the map

$$G \ni g \longrightarrow \alpha_g(y) \in A(X)$$

is a smooth map from G to the normed vector space $A(X)$. Then

$$\mathcal{A}^\infty(X) = \{y \in A(X) \,|\, y \text{ is a smooth vector}\}$$

is a $*$-subalgebra of $A(X)$ which is norm dense in $A(X)$. Since G acts *fibrewise* on $A(X)$, it follows that $\mathcal{A}^\infty(X)$ is again a $C_0(X)$-algebra which is *fibrewise smooth*.

Let T denote the torus of dimension n. We define a *H_3-twisted noncommutative principal T-bundle* (or *H_3-twisted NCP T-bundle*) over X to be a separable C^*-bundle $A(X)$ together with a fibrewise action $\alpha : T \to \mathrm{Aut}(A(X))$ such that there is a Morita equivalence,

$$A(X) \rtimes_\alpha T \cong \mathrm{C}T(X, H_3),$$

as C^*-bundles over X, where $\mathrm{C}T(X, H_3)$ denotes the continuous trace C^*-algebra with spectrum equal to X and Dixmier-Douady class equal to $H_3 \in H^3(X, \mathbf{Z})$.

If $A(X)$ is a H_3-twisted NCP T-bundle over X, then we call $\mathcal{A}^\infty(X)$ a *fibrewise smooth H_3-twisted noncommutative principal T-bundle* (or *fibrewise smooth H_3-twisted NCP T-bundle*) over X. In this paper, we are able to give a complete classification of fibrewise smooth H_3-twisted NCP T-bundles over X via a parametrised version of Rieffel's theory of strict deformation quantization as derived in [9].

2. Parametrised strict deformation quantization of C^*-bundles with fibrewise action of T

In a nutshell, parametrised strict deformation quantization is a functorial extension of Rieffel's strict deformation quantization from algebras A, to $C(X)$-algebras $A(X)$, and in particular to C^*-bundles over X. Unlike Rieffel's deformation theory [18] the version in [9] starts with multipliers via the Landstad–Kasprzak approach. Here we review the theory for C^*-bundles over X.

Let $A(X)$ be a C^*-algebra bundle over X with a fibrewise action α of a torus T. Let $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. Then we define the parametrised strict deformation quantization of $A(X)$,

denoted $A(X)_\sigma$ as follows. We have the direct sum decomposition,

$$\begin{array}{rcl} A(X) & \cong & \widehat{\bigoplus}_{\chi\in\hat{T}} A(X)_\chi \\ \phi(x) & = & \sum_{\chi\in\hat{T}} \phi_\chi(x) \end{array}$$

for $x \in X$, where for $\chi \in \widehat{T}$,

$$A(X)_\chi := \{a \in A(X) \mid \alpha_t(a) = \chi(t)\cdot a \quad \forall\, t \in T\}\,.$$

Since T acts by $\star$-automorphisms, we have

$$A(X)_\chi \cdot A(X)_\eta \subseteq A(X)_{\chi\eta} \quad \text{and} \quad A(X)^*_\chi = A(X)_{\chi^{-1}} \qquad \forall\, \chi, \eta \in \widehat{T}. \tag{1}$$

Therefore the spaces $A(X)_\chi$ for $\chi \in \widehat{T}$ form a Fell bundle $A(X)$ over $\widehat{T}$ (see [8]); there is no continuity condition because $\widehat{T}$ is discrete. The completion of the direct sum is explained as follows. The representation theory of T shows that $\bigoplus_{\chi\in\widehat{T}} A(X)_\chi = A(X)^{alg}$ is a T-equivariant dense subspace of $A(X)$, where T acts on $A(X)_\chi$ as follows: $\hat{\alpha}_t(\phi_\chi(x)) = \chi(t)\phi_\chi(x)$ for all $t \in T$, $x \in X$. Then $\widehat{A(X)^{alg}} = \widehat{\bigoplus}_{\chi\in\hat{T}} A(X)_\chi$ is the completion in the C^*-norm of $A(X)$, and is isomorphic to $A(X)$.

The product of elements in $A(X)^{alg} \subset A(X)$ then also decompose as,

$$(\phi\psi)_\chi(x) = \sum_{\chi_1\chi_2=\chi} \phi_{\chi_1}(x)\psi_{\chi_2}(x)$$

for $\chi_1, \chi_2, \chi \in \widehat{T}$. The product can be deformed by setting

$$(\phi \star_\sigma \psi)_\chi(x) = \sum_{\chi_1\chi_2=\chi} \phi_{\chi_1}(x)\psi_{\chi_2}(x)\sigma(x;\chi_1,\chi_2)$$

which is associative because of the cocycle property of σ.

We next describe the norm completion aspects. For $x \in X$, let $\mathcal{H}_x$ denote the universal Hilbert space representation of the fibre C^*-algebra $A(X)_x$ which one obtains via the GNS theorem. Let $\mathcal{H}_1 = \int_X \mathcal{H}_x dx$ denote the direct integral, which is the universal Hilbert space representation of $A(X)$. By considering instead the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes L^2(T) \otimes \mathcal{H}_2$, where $\mathcal{H}_2$ is an infinite dimensional Hilbert space, where we note that every character of T occurs with infinite multiplicity in $L^2(T) \otimes \mathcal{H}_2$, we obtain a T-equivariant embedding $\varpi : A(X) \to B(\mathcal{H})$. The equivariance means that

$$\varpi(\phi(x)_\chi) = \varpi(\phi(x))_\chi.$$

Now consider the action of $A(X)^{alg}$ on $\mathcal{H}$ given by the deformed product $\star_\sigma$, that is, for $\phi \in A(X)^{alg}$ and $\Psi \in \mathcal{H}$,

$$(\phi \star_\sigma \Psi)_\chi(x) = \sum_{\chi_1\chi_2=\chi} \varpi(\phi_{\chi_1}(x))\Psi_{\chi_2}(x)\sigma(x;\chi_1,\chi_2).$$

The operator norm completion of this action is the parametrised strict deformation quantization of $A(X)$, denoted by $A(X)_\sigma$.

We next consider a special case of this construction. Consider a smooth fiber bundle of smooth manifolds,

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & & \downarrow \pi \\ & & X. \end{array} \tag{2}$$

Suppose there is a fibrewise action of a torus T on Y. That is, assume that there is an action of T on Y satisfying,

$$\pi(t.y) = \pi(y), \qquad \forall\, t \in T,\ y \in Y.$$

Let $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. $C_0(Y)$ is a C^*-bundle over X, and as above, form the parametrised strict deformation quantization $C_0(Y)_\sigma$.

In particular, let Y be a principal G-bundle over X, where G is a compact Lie group such that $\mathrm{r}ank(G) \geq 2$. (e.g. $G = \mathrm{S}U(n)$, $n \geq 3$ or $G = \mathrm{U}(n)$, $n \geq 2$). Let T be a maximal torus in G and $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. Then $C_0(Y)$ is a C^*-bundle over X, and as above, form the parametrised strict deformation quantization $C_0(Y)_\sigma$.

3. Classifying H_3-twisted NCPT-bundles

Here we prove another application of parametrised strict deformation quantization cf. §2, [9].

Let $A(X)$ be a H_3-twisted NCPT-bundle over X. Consider the C^*-bundle over X, $A(X) \otimes_{C_0(X)} \mathrm{C}T(X, -H_3)$, which has a fibrewise (diagonal) action of T, where T acts trivially on $\mathrm{C}T(X, -H_3)$. Then

$$\begin{aligned} \big(A(X) \otimes_{C_0(X)} \mathrm{C}T(X, -H_3)\big) \rtimes T &\cong (A(X) \rtimes T) \otimes_{C_0(X)} \mathrm{C}T(X, -H_3) \\ &\cong \mathrm{C}T(X, H_3) \otimes_{C_0(X)} \mathrm{C}T(X, -H_3) \\ &\cong C_0(X, \mathcal{K}). \end{aligned}$$

Therefore, $A(X) \otimes_{C_0(X)} \mathrm{C}T(X, -H_3)$ is a NCPT-bundle over X. By the classification Theorem 5.1 [9], (which in turn used the results of Echterhoff and Williams, [7]) we deduce that

$$A(X) \otimes_{C_0(X)} \mathrm{C}T(X, -H_3) \cong C_0(Y)_\sigma.$$

Therefore,

$$A(X) \cong C_0(Y)_\sigma \otimes_{C_0(X)} \mathrm{C}T(X, H_3).$$

Lemma 3.1. *In the notation above,*

$$C_0(Y)_\sigma \otimes_{C_0(X)} CT(X, H_3) \cong CT(Y, q^*(H_3))_\sigma$$

Proof. We use the explicit Fourier decomposition as in Example 5.2 and Example 6.2 [9] to deduce that first of all that both sides are naturally isomorphic as T-vector spaces, and also that products are compatible under the isomorphism. $\square$

To summarize, we have the following main result, which follows from Theorem 3.1 [9], §4 [9], Example 5.2, and the observations above.

Theorem 3.2. [1] *Given a H_3-twisted NCPT-bundle $A(X)$, there is a defining deformation $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ and a principal torus bundle $q : Y \to X$ such that $A(X)$ is T-equivariant Morita equivalent over $C_0(X)$, to the parametrised strict deformation quantization of $CT(Y, q^*(H_3))$ with respect to σ, that is,*

$$A(X) \cong CT(Y, q^*(H_3))_\sigma.$$

Conversely, by Example 5.2, the parametrised strict deformation quantization of $CT(Y, q^*(H_3))$ is the H_3-twisted NCPT-bundle $CT(Y, q^*(H_3))_\sigma$.

4. T-duality and K-theory

Here we prove the result,

Theorem 4.1. *In the notation above, $(X \times T, H_1 + H_2 + H_3)$ and the parametrised strict deformation quantization of $(Y, q^*(H_3))$ with deformation parameter σ, $[\sigma] = H_1$, are T-dual pairs, where the 1st Chern class $c_1(Y) = H_2$. That is,*

$$CT(Y, q^*(H_3))_\sigma \rtimes V \cong CT(X \times T, H_1 + H_2 + H_3).$$

Proof. As before, let V be the vector group that is the universal covering group of the torus group T, and the action of V on the spectrum factors through T. By Lemma 8.1 in [6], the crossed product $\mathrm{C}T(X \times T, H_1 + H_2) \rtimes_\beta V$ is Morita equivalent to $C_0(X, \mathcal{K}) \rtimes_\sigma \widehat{T}$, where as before, the Pontryagin dual group $\widehat{T}$ acts fibrewise on $C_0(X, \mathcal{K})$. Setting $A(X) = C_0(X, \mathcal{K}) \rtimes_\sigma \widehat{T}$, then it is a C^*-bundle over X with a fibrewise action of T and by Takai duality, $A(X) \rtimes T \cong C_0(X, \mathcal{K})$. Therefore $A(X)$ is a NCPT-bundle and by Theorem 5.1 in [9], we see that there is a T-equivariant Morita equivalence,

$$A(X) \sim C_0(Y)_\sigma,$$

where the notation is as in the statement of this Theorem. By Lemma 3.1,

$$\mathrm{C}T(Y, q^*(H_3))_\sigma \rtimes V \cong (C_0(Y)_\sigma \rtimes V) \otimes_{C_0(X)} \mathrm{C}T(X, H_3).$$

By Takai duality, $C_0(Y)_\sigma \rtimes V \cong \mathrm{C}T(X \times T, H_1 + H_2)$. Therefore

$$\mathrm{C}T(Y, q^*(H_3))_\sigma \rtimes V \cong \mathrm{C}T(X \times T, H_1 + H_2 + H_3),$$

proving the result. □

Using Connes Thom isomorphism theorem [4] and the result above, one has

[1] This also easily follows from the well known fact that in the Brauer group $\mathrm{B}r_G(X)$ for G acting trivially on X, every element in this Brauer group factors by a product of the trivial action on A and an action on $C_0(X, \mathcal{K})$ (e.g. see [5]). Anyway, the arguments presented here are direct and quite simple.

Corollary 4.2. *The K-theory of $CT(Y, q^*(H_3))_\sigma$ depends on the deformation parameter in general. More precisely, in the notation above $[\sigma] = H_1$, $c_1(Y) = H_2$,*

$$K_\bullet(CT(Y, q^*(H_3))_\sigma) \cong K^{\bullet + \dim V}(X \times T, H_1 + H_2 + H_3),$$

where the right hand side denotes the twisted K-theory.

5. Examples

Example 5.1 (Noncommutative torus)**.** We begin by recalling the construction by Rieffel [18] realizing the smooth noncommutative torus as a deformation quantization of the smooth functions on a torus $T = \mathbf{R}^n/\mathbf{Z}^n$ of dimension equal to n.

Recall that any translation invariant Poisson bracket on T is just

$$\{a, b\} = \sum \theta_{ij} \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_j},$$

for $a, b \in C^\infty(T)$, where (θ_{ij}) is a skew symmetric matrix. The action of T on itself is given by translation. The Fourier transform is an isomorphism between $C^\infty(T)$ and $\mathcal{S}(\hat{T})$, taking the pointwise product on $C^\infty(T)$ to the convolution product on $\mathcal{S}(\hat{T})$ and taking differentiation with respect to a coordinate function to multiplication by the dual coordinate. In particular, the Fourier transform of the Poisson bracket gives rise to an operation on $\mathcal{S}(\hat{T})$ denoted the same. For $\phi, \psi \in \mathcal{S}(\hat{T})$, define

$$\{\psi, \phi\}(p) = -4\pi^2 \sum_{p_1 + p_2 = p} \psi(p_1)\phi(p_2)\gamma(p_1, p_2)$$

where γ is the skew symmetric form on $\hat{T}$ defined by

$$\gamma(p_1, p_2) = \sum \theta_{ij}\, p_{1,i}\, p_{2,j}.$$

For $\hbar \in \mathbf{R}$, define a skew bicharacter $\sigma_\hbar$ on $\hat{T}$ by

$$\sigma_\hbar(p_1, p_2) = \exp(-\pi\hbar\gamma(p_1, p_2)).$$

Using this, define a new associative product $\star_\hbar$ on $\mathcal{S}(\hat{T})$,

$$(\psi \star_\hbar \phi)(p) = \sum_{p_1 + p_2 = p} \psi(p_1)\phi(p_2)\sigma_\hbar(p_1, p_2).$$

This is precisely the smooth noncommutative torus $A^\infty_{\sigma_\hbar}$.

The norm $|| \cdot ||_\hbar$ is defined to be the operator norm for the action of $\mathcal{S}(\hat{T})$ on $L^2(\hat{T})$ given by $\star_\hbar$. Via the Fourier transform, carry this structure back to $C^\infty(T)$, to obtain the smooth noncommutative torus as a strict deformation quantization of $C^\infty(T)$, [18] with respect to the translation action of T.

Example 5.2. We next generalize the first example above to the case of principal torus bundles $q : Y \to X$ of rank equal to n, together with a algebra bundle $\mathcal{K}_P \to X$ with fibre the C^*-algebra of compact operators

$\mathcal{K}$. Here $P \to X$ is a principal bundle with structure group the projective unitary group PU, which acts on $\mathcal{K}$ by conjugation. Note that fibrewise smooth sections of $q^*(\mathcal{K}_P)$ over Y decompose as a direct sum,

$$\begin{array}{rcl} C^\infty_{\rm fibre}(Y, q^*(\mathcal{K}_P)) & = & \widehat{\bigoplus}_{\alpha\in\hat{T}}\, C^\infty_{\rm fibre}(X, \mathcal{L}_\alpha \otimes \mathcal{K}_P) \\ \phi & = & \sum_{\alpha\in\hat{T}} \phi_\alpha \end{array}$$

where $C^\infty_{\rm fibre}(X, \mathcal{L}_\alpha \otimes E)$ is defined as the subspace of $C^\infty_{\rm fibre}(Y, q^*(E))$ consisting of sections which transform under the character $\alpha \in \hat{T}$, and where $\mathcal{L}_\alpha$ denotes the associated line bundle $Y \times_\alpha \mathbf{C}$ over X. That is, $\phi_\alpha(yt) = \alpha(t)\phi_\alpha(y), \forall\, y \in Y, t \in T$. The direct sum is completed in such a way that the function $\hat{T} \ni \alpha \mapsto ||\phi_\alpha||_\infty \in \mathbf{R}$ is in $\mathcal{S}(\hat{T})$.

For $\phi, \psi \in C^\infty_{\rm fibre}(Y, q^*(\mathcal{K}_P))$, define a deformed product $\star_\hbar$ as follows. For $y \in Y$, $\alpha, \alpha_1, \alpha_2 \in \hat{T}$, let

$$(\psi \star_\hbar \phi)(y, \alpha) = \sum_{\alpha_1\alpha_2=\alpha} \psi(y, \alpha_1)\phi(y, \alpha_2)\sigma_\hbar(q(y); \alpha_1, \alpha_2), \tag{3}$$

using the notation $\psi(y, \alpha_1) = \psi_{\alpha_1}(y)$ etc., and where $\sigma_\hbar \in C(X, Z^2(\hat{T}, \mathbf{T}))$ is a continuous family of bicharacters of $\hat{T}$ such that $\sigma_0 = 1$. The cocycle property of $\sigma_\hbar$ ensures that (3) defines an associative product. The construction is clearly T-equivariant. We denote the deformed algebra as $C^\infty_{\rm fibre}(Y, q^*(\mathcal{K}_P))_{\sigma_\hbar}$.

6. General torus bundles

We extend the results of the previous sections to the case of deformations of general torus bundles, not just principal torus bundles. That is, the noncommutative torus bundles (NCT-bundles) of this section strictly include the noncommutative principal torus bundles (NCPT-bundles) of [9],[6].

For any fibre bundle $F \to\; Y \xrightarrow{\xi} X$ with structure group G, the action of G on F induces an action of $\pi_0(G)$ on the homology and cohomology of F. When X is a connected manifold, there is a well-defined homomorphism $\pi_1(X) \to \pi_0(G)$ that gives each homology or cohomology group of F the structure of a $\mathbf{Z}\pi_1(X)$-module. Now suppose that F is a torus T and $G = \mathrm{D}iff(T)$. It is well known that $\pi_0(G) \cong \mathrm{G}L(n, \mathbf{Z})$, where $n = \dim(T)$, and that the $\pi_0(G)$-action on $H_1(T)$ may be identified with the natural action of $\mathrm{G}L(n, \mathbf{Z})$ on $\widehat{T}$. Given any representation $\rho : \pi_1(X) \to \mathrm{G}L(n, \mathbf{Z})$, we let $\mathbf{Z}^n_\rho$ denote the corresponding $\mathbf{Z}\pi_1(B)$-module. We will make use of the following proposition, which is well known, and explicitly stated in the appendices of [10].

Proposition 6.1. *Assume that X is a compact, connected manifold, and choose any representation ρ : $\pi_1(X) \to GL(n, \mathbf{Z})$. Then there is a natural, bijective correspondence between the equivalence classes of torus bundles over X inducing the module structure $\mathbf{Z}^n_\rho$ on $H_1(T)$ and the elements of $H^2(X; \mathbf{Z}^n_\rho)$, the second cohomology group of X with local coefficients $\mathbf{Z}^n_\rho$.* □

Remark 6.2. We call the cohomology class corresponding to the symplectic torus bundle ξ the *characteristic class* of ξ and denote it by $c(\xi) \in H^2(X; \mathbf{Z}^n_\rho)$. Then the characteristic class $c(\xi)$ vanishes if and only if ξ admits a section. When the representation ρ is trivial, then ξ is a principal torus bundle and the characteristic class reduces to the first Chern class. $c(\xi) = c_1(\xi) = 0$ if and only if ξ is trivial.

Let X be compact and $T \to Y \overset{\xi}{\to} X$ be a torus bundle over X. Let $\Gamma \to \widehat{X} \overset{\eta}{\to} X$ denote the universal cover of X, and consider the lifted torus bundle $T \to \eta^*(Y) \overset{\eta^*\xi}{\to} \widehat{X}$. Since $\widehat{X}$ is simply-connected, it is classified by a characteristic class in $H^2(\widehat{X}; \mathbf{Z}^n)$, so that $T \to \eta^*(Y) \overset{\eta^*\xi}{\to} \widehat{X}$ is a principal torus bundle. Let $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. Then $C_0(\eta^*(Y))$ is a C^*-bundle over $\widehat{X}$, and as before, form the parametrised strict deformation quantization $C_0(\eta^*(Y))_{\widehat{\sigma}}$, where $\widehat{\sigma}$ is the lift of σ to $\widehat{X}$. Since $\eta^*(Y)$ is the total space of a principal Γ-bundle over the compact space Y, the action of Γ on $\eta^*(Y)$ is free and proper, so the action of Γ on $C_0(\eta^*(Y))_{\widehat{\sigma}}$ is also proper in the sense of Rieffel [21]. In particular, the fixed-point algebra $C_0(\eta^*(Y))^{\Gamma}_{\widehat{\sigma}}$ makes sense and is the parametrised strict deformation quantization of $C_0(Y)$. We define this to be a *noncommutative torus bundle* (NCT-bundle) over X. In particular, they strictly include noncommutative principal torus bundles (NCPT-bundles) over X. NCT-bundles are classified by a representation $\rho : \pi_1(X) \to \mathrm{G}L(n, \mathbf{Z})$ together with a cohomology class with local coefficients, $H_2 \in H^2(X; \mathbf{Z}^n_\rho)$ and a deformation parameter $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$.

Finally, we define a H_3-twisted NCT-bundle over X to be a parametrised strict deformation quantization of $\mathrm{C}T(Y, \xi^*(H_3))$, denoted by $\mathrm{C}T(\eta^*(Y), \eta^*\xi^*(H_3))^{\Gamma}_{\widehat{\sigma}}$.

7. Deformations of monoidal categories

Rieffel's strict deformation theory modifies the multiplication on an algebra. As noted in the introduction it is a useful principle that deformation should involve not only an algebra, but a category in which the algebra is but one object. Fortunately Rieffel's strict C*-algebra deformation can be extended to give a functor changing the tensor product in a monoidal category of Fréchet V-modules for a vector group V. Another motivation for this stems from the way in which nonassociative crossed products could be understood naturally in the context of a monoidal category, [1]. A similar interpretation of noncommutative crossed products could make it easier to unify the two examples. Let X be a locally compact Hausdorff space and V an abelian group. We start with Banach V-modules (in which V acts by isometries), with a commuting action of $C_0(X)$. Each module has a dense submodule of V-smooth vectors, on which V and $C_0(X)$ still act and which can be given the structure of a Fréchet space. Consider the strict symmetric monoidal category of these $C_0(X)$-V-mod of smooth $C_0(X) \rtimes V$-modules with tensor product $\otimes_0 = \otimes_{C_0(X)}$ as in [22], and unit object $C_0(X)$ (with the multiplication action of $C_0(X)$ and trivial action of V). We shall suppose also that we have a symmetric bicharacter $e = e^{iB} : V \times V \to C_0(X)$, and a B-skew-adjoint automorphism J of V. The monoidal functor $\mathcal{D}_J$ to a braided category $C_0(X) - (V, J)$-mod acts as the identity on objects and morphisms, but gives a braided tensor product $\otimes_J$ and (assuming the integral well-defined) with, for objects $\mathcal{A}$ and $\mathcal{B}$, the consistency map $c_J : \mathcal{A} \otimes_J \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$, taking $x \in \mathcal{A}$, $y \in \mathcal{B}$

$$c_J(x \otimes_J y) = \int_{V \times V} e(u, v)((Ju).x) \otimes_0 (v.y)\, du\, dv,$$

where e acts on the tensor product as an element of $C_0(X)$. (It follows from the discussion in [18, 9] that c_J is invertible.) Essentially all the technical estimates needed to show that this is well-defined, and to derive its

properties have already been provided by Rieffel in [18, Chapter I] using a simplified version of Hörmander's partial integration technique applied to smooth maps from a vector group to a Fréchet space. In a Hilbert V module M the smooth vectors M^∞ for the action provide a Fréchet space, and the rest is done as in [18]. (Within the tensor product of two V-modules M_1 and M_2 is the dense subspace of smooth vectors $(M_1 \otimes M_2)^\infty$, which contains the tensor product $M_1^\infty \otimes M_2^\infty$. We have just mimicked the constructions of [18, Chapter 2] to deform the tensor product, rather than an algebra product.) One can check that this is $C_0(X)$-linear (due to the triviality of the action of V on $C_0(X)$) and is compatible with strict associativity [18, Theorem 2.14], and $C_0(X)$ as unit object. (For non-vanishing H_0, we would instead need consistency with the new associativity map in the usual hexagonal diagram.) Since V is abelian it has the tensor product action Δ of V, (which changes to $c_J^{-1} \circ \Delta \circ c_J$in the new monoidal category). Some care is needed because the asymmetry in c_J means that deformed category is braided. Assuming that one started with a trivial braiding given by the flip $\Psi_0 : m \otimes_0 n \to n \otimes_0 m$, one obtains $\Psi_J = c_J^{-1} \Psi_0 c_J : M \otimes_J N \to N \otimes_J M$. Since $\Psi_0^2 = 1$, we automatically have $\Psi_J^2 = 1$, so that Ψ_J is a symmetric braiding. More generally Ψ_J and Ψ_0 satisfy the same polynomial identities: if Ψ_0 is of Hecke type then so is Ψ_J. If an object $\mathcal{A}$ has a multiplication morphism $\mu : \mathcal{A} \otimes_0 \mathcal{A} \to \mathcal{A}$, then the morphism property ensures that V acts by automorphisms, and the deformed multiplication is $\mu \circ c_J : \mathcal{A} \otimes_J \mathcal{A} \to \mathcal{A}$, or, using $\star_J$ and $\star$ for the deformed and undeformed multiplications,

$$x \star_J y = \int_{V \times V} e(u, v)((Ju).x) \star (v.y)\, du\, dv,$$

which is the Rieffel deformed product [18]. Due to the braiding a commutative algebra product μ (satisfying $\mu \circ \Psi = \mu$) turns into a braided commutative, but noncommutative, product with $\mu_J \circ \Psi_J = \mu_J$, as one would expect for a map transforming classical to quantum theory. We can similarly deform $\mathcal{A}$-modules and bimodules. For example, an action $\alpha : \mathcal{A} \otimes_0 M \to M$ can be deformed to $\alpha_J : \mathcal{A} \otimes_J M \to M$

$$\alpha_J(a)[m] = \int_{V \times V} e(u, v)\alpha((Ju).a)[v.m]\, du\, dv.$$

One can follow the same strategy as in [1] and study the effects on compact operators, crossed products etc, but we shall content ourselves with a discussion of the modules. As in [18, Theorem 2.15] we can show that the functors for different J satisfy $\mathcal{D}_K \circ \mathcal{D}_J = \mathcal{D}_{J+K}$.

8. Parametrised strict deformation quantization of Hilbert C^*-modules over C^*-bundles

As an application of the general procedure outlined in §7, we extend the parametrised strict deformation quantization of C^*-bundles $A(X)$ to Hilbert C^*-modules over $A(X)$, but somewhat more directly.

Recall that Hilbert C^*-modules generalise the notion of a Hilbert space, in that they endow a linear space with an inner product which takes values in a C^*-algebra. They were developed by Marc Rieffel in [22], which used Hilbert C^*-modules to construct a theory of induced representations of C^*-algebras. In [20], Rieffel used Hilbert C^*-modules to extend the notion of Morita equivalence to C*-algebras. In [11],

Kasparov used Hilbert C^*-modules in his formulation of bivariant K-theory. Hermitian vector bundles are examples of Hilbert C^*-modules over commutative C^*-algebras.

Let $A(X)$ be a C^*-algebra bundle over X, and $E(X)$ a Hilbert C^*-module over $A(X)$ respecting the fibre structure. That is, $E(X)$ has an $A(X)$-valued inner product,

$$\langle .,. \rangle : E(X) \times E(X) \longrightarrow A(X),$$

and the fibre $E(X)_x : E(X)/I_x$ is a (left) Hilbert C^*-module over $A(X)_x$ for all $x \in X$.

Suppose now that $A(X)$ has a fibrewise action α of a torus T and let $E(X)$ have a compatible fibrewise action of T.

We have the direct sum decomposition,

$$\begin{array}{rcl} E(X) & \cong & \widehat{\bigoplus}_{\chi \in \hat{T}} E(X)_\chi \\ \psi(x) & = & \sum_{\chi \in \hat{T}} \psi_\chi(x) \end{array}$$

for $x \in X$, where for $\chi \in \widehat{T}$,

$$E(X)_\chi := \{\psi \in E(X) \mid \alpha_t(\psi) = \chi(t) \cdot \psi \quad \forall\, t \in T\}\,.$$

Since T acts by $\star$-automorphisms, we have

$$A(X)_\chi \cdot E(X)_\eta \subseteq E(X)_{\chi\eta} \quad and \quad E(X)^*_\chi = E(X)_{\chi^{-1}} \qquad \forall\, \chi, \eta \in \widehat{T}. \tag{4}$$

Therefore the spaces $E(X)_\chi$ for $\chi \in \widehat{T}$ form a Fell bundle $E(X)$ over $\widehat{T}$ (see [8]); there is no continuity condition because $\widehat{T}$ is discrete. The completion of the direct sum is explained as before. The representation theory of T shows that $\bigoplus_{\chi \in \hat{T}} E(X)_\chi$ is a T-equivariant dense subspace of $E(X)$, where T acts on $E(X)_\chi$ as follows: $\hat{\alpha}_t(\phi_\chi(x)) = \chi(t)\phi_\chi(x)$ for all $t \in T$, $x \in X$. Then $\widehat{\bigoplus}_{\chi \in \hat{T}} E(X)_\chi$ is the completion in the Hilbert C^*-norm of $E(X)$.

The action of $A(X)$ on $E(X)$ then also decomposes as,

$$(\phi\psi)_\chi(x) = \sum_{\chi_1\chi_2 = \chi} \phi_{\chi_1}(x)\psi_{\chi_2}(x)$$

for $\chi_1, \chi_2, \chi \in \widehat{T}$, $\phi \in A(X)$ and $\psi \in E(X)$.

Let $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. Then as in section 2, the parametrised strict deformation quantization of $A(X)$ can be defined, and is denoted by $A(X)_\sigma$.

The action of $A(X)_\sigma$ on $E(X)$ can be defined by deforming the action of $A(X)$ on $E(X)$,

$$(\phi \star_\sigma \psi)_\chi(x) = \sum_{\chi_1\chi_2 = \chi} \phi_{\chi_1}(x)\psi_{\chi_2}(x)\sigma(x; \chi_1, \chi_2)$$

for $\chi_1, \chi_2, \chi \in \widehat{T}$, $\phi \in A(X)_\sigma$ and $\psi \in E(X)$. This is an action because $\sigma(x :, \cdot, \cdot)$ is a 2 cocycle for all $x \in X$.

This is the parametrised strict deformation quantization of $E(X)$, denoted by $E(X)_\sigma$, which is a Hilbert C^*-module over $A(X)_\sigma$.

Example 8.1. Let $E \to X$ be a complex vector bundle over X. Consider a smooth fiber bundle of smooth manifolds,

$$
\begin{tikzcd}
Z \arrow[r] & Y \arrow[d, "\pi"] \\
& X.
\end{tikzcd}
\tag{5}
$$

Suppose there is a fibrewise action of a torus T on Y. That is, assume that there is an action of T on Y satisfying,

$$\pi(t.y) = \pi(y), \qquad \forall\, t \in T,\, y \in Y.$$

Let $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. $C_0(Y)$ is a C^*-bundle over X, and as in section 2, form the parametrised strict deformation quantization $C_0(Y)_\sigma$.

Then sections $C_0(Y, \pi^*(E))$ is a Hilbert C^*-module over the C^*-bundle $C_0(Y)$ over X, with a fibrewise T-action compatible with the fibrewise action of T on $C_0(Y)$. Therefore as above, we can construct a parametrised strict deformation quantization of $C_0(Y, \pi^*(E))$, denoted by $C_0(Y, \pi^*(E))_\sigma$, which is a Hilbert C^*-module over $C_0(Y)_\sigma$.

Acknowledgements. V.M. thanks the Australian Research Council for support, and also thanks the hospitality at Oxford University where part of this research was completed.

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